A Level-Set Framework for Shape Optimisation

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IGDK 1754

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Overview

The Level-Set Method

The Level-Set Function

Geometries as **level sets** of $\phi : D \subset \mathbb{R}^n \to \mathbb{R}$:

$$\begin{aligned} \Omega = \Omega(\phi) &= \{x \in D \mid \phi(x) < 0\} \\ \Gamma = \partial\Omega &= \{x \in D \mid \phi(x) = 0\} \end{aligned}$$

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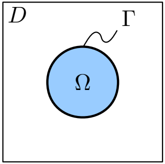
Also "irregular" shapes are possible:

$$\phi(x, y) = \sqrt{x^2 + y^2} - 1 \quad \text{or} \quad \phi(x, y) = x^2 + y^2 - 1$$



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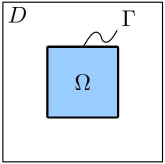
$$\phi(x, y) = \max\left(|x| - 1, |y| - 1\right)$$



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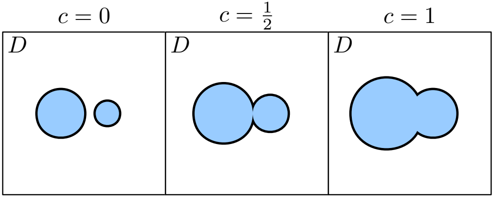
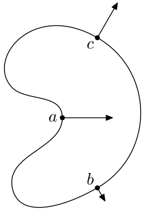
Also “irregular” shapes are possible:

$$\phi(x, y) = \min\left(\sqrt{(x-2)^2 + y^2} - 1, \sqrt{(x+2)^2 + y^2} - 2\right) - c$$



The Speed Method in Normal Direction

Speed field $F : D \to \mathbb{R}$ in **normal direction**:



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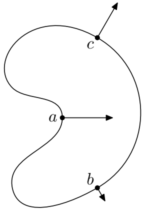
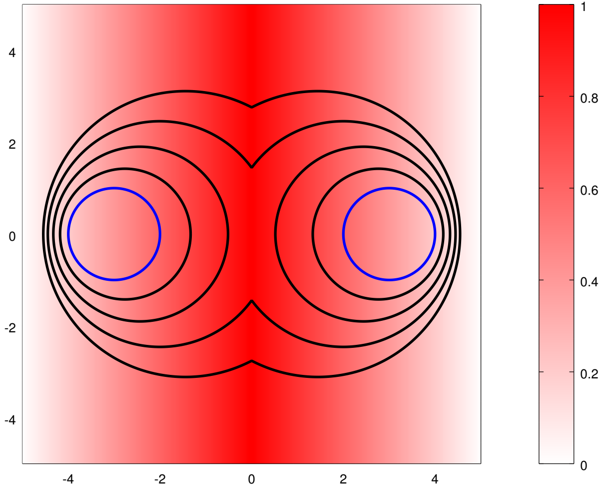


Level-Set Equation

$$\phi_t(x,t) + F(x)\,|\nabla\phi(x,t)| = 0, \quad \phi_0(x,0) = \phi_0(x)$$

It has a **unique viscosity solution**, see Crandall, Ishii, Lions [3] and Giga [5]. Original work by Osher, Sethian [6].

Changes in Topology Are Possible



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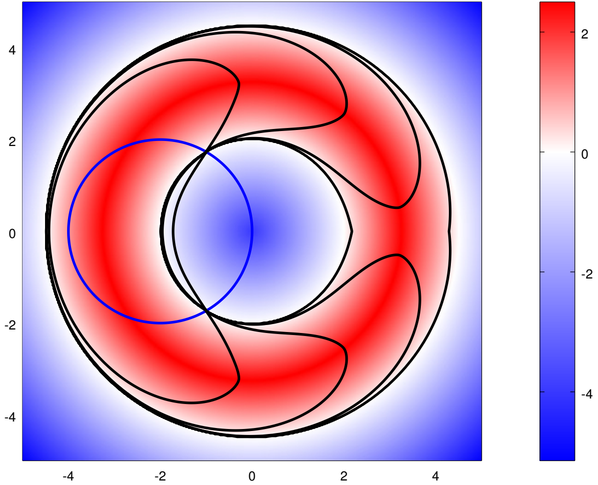

The Sign of F

Important is the **sign of F**:

Theorem

Let $F = F^+ - F^-$ be the decomposition in positive and negative parts, and

$$\phi_t^\pm(x,t) + F^\pm(x)\left|\nabla\phi^\pm(x,t)\right| = 0, \quad \phi^\pm(x,0) = \pm\phi_0(x).$$

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Then:

$$\phi(x,t) = \begin{cases} \phi^+(x,t) & F(x) > 0 \\ \phi_0(x) & F(x) = 0 \\ -\phi^-(x,t) & F(x) < 0 \end{cases}$$

$F(x) = 0$ means $\phi(x,\cdot)$ is constant: $\rightarrow$ **Geometric constraints!**

One can reduce all considerations to the case $F \geq 0$.

Mayer's Problem

Paths suited to $F \geq 0$:

$$S_t(x) = \{\xi \in W^{1,\infty}([0,t]) \mid \xi(0) = x,\ |\xi'(\tau)| \leq F(\xi(\tau)) \text{ for all } \tau \in [0,t]\}$$

Reachable set:

$$R_t(x) = \{\xi(t) \mid \xi \in S_t(x)\}$$

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$$V(x,t) = \inf_{\xi \in S_t(x)} \phi_0(\xi(t)) = \inf_{y \in R_t(x)} \phi_0(y)$$

Hamilton-Jacobi-Bellman: The level-set equation!

The Hopf-Lax Formula

Optimal-control theory implies a **Hopf-Lax Formula**:

Let d solve the **Eikonal equation** for the speed F:

$$F(x)\,|\nabla d(x)| = 1$$

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Theorem (Hopf-Lax Formula)

Let $F \geq 0$ be Lipschitz continuous and have compact support in D. For all x with $F(x) > 0$ and $\phi_0(x) > 0$, the solution of the level-set equation is given as:

$$\phi(x,t) = \inf\left\{\phi_0(y) \mid d(x,y) \leq t\right\}$$

See also Falcone, Giorgi, Loreti [4] and Capuzzo-Dolcetta [2].

Representation of the Level-Set Domain

Distance to Initial Geometry

$$d_0(x) = \inf_{y \in \Omega_0} d(x, y)$$

→ **When does the advancing front hit x?**

Efficient computation possible with **Fast Marching** (Sethian [7]).

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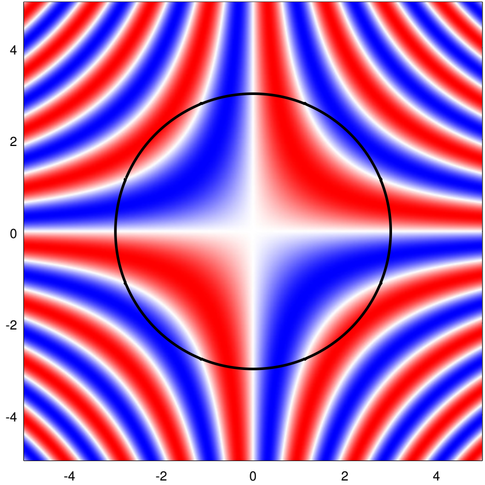
Theorem (Representation Formula)

Let $F \geq 0$. The time evolution of Ω_0 is given by:

$$\begin{aligned} \Omega_t &= \{x \in D \mid d_0(x) < t\} \\ \Gamma_t &= \{x \in D \mid d_0(x) = t\} \end{aligned}$$

Generalisation is possible for arbitrary signs of F:
→ **"Composite Fast Marching"**

Demonstration



Non-Fattening

Classical result for $\Gamma_t = \partial\Omega_t$: Non-fattening in a **topological sense**, considering the **space-time**. (See Barles, Soner, Souganidis [1].)

Based on our formula, one can easily deduce:

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Theorem (Topological Non-Fattening)

Let $F \geq 0$ *and* $\overline{\Omega_0} = \Omega_0 \cup \Gamma_0$. *Then* $\overline{\Omega_t} = \Omega_t \cup \Gamma_t$ *for all* $t \geq 0$.
If $F \leq 0$ *and* $(\Omega_0 \cup \Gamma_0)^\circ = \Omega_0$, *then* $(\Omega_t \cup \Gamma_t)^\circ = \Omega_t$ *for all* $t \geq 0$.

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Theorem (Topological Non-Fattening)

Let $F \geq 0$ and $\overline{\Omega_0} = \Omega_0 \cup \Gamma_0$. Then $\overline{\Omega_t} = \Omega_t \cup \Gamma_t$ for all $t \geq 0$.
If $F \leq 0$ and $(\Omega_0 \cup \Gamma_0)^\circ = \Omega_0$, then $(\Omega_t \cup \Gamma_t)^\circ = \Omega_t$ for all $t \geq 0$.

Theorem (Measure-Theoretic Non-Fattening)

Let $|\Gamma_0| = 0$. Then $|\Gamma_t| = 0$ for all $t \geq 0$.

Gradient-Descent Methods

Shape Calculus

Theorem

Let $f \in L^1_{loc}(D)$ *and assume* $F \geq 0$. *Then:*

$$J(t) = \int_{\Omega_t} f \, dx = \int_{\Omega_0} f \, dx + \int_0^t \int_{\Gamma_s} Ff \, d\sigma \, ds$$

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$$J(t) = \int_{\Omega_t} f \, dx = \int_{\Omega_0} f \, dx + \int_0^t \int_{\Gamma_s} Ff \, d\sigma \, ds$$

This yields the **shape derivative** of J:

$$dJ(\Omega_t; F) = J'(t) = \int_{\Gamma_t} Ff \, d\sigma$$

(In a **weak sense**, since J is **absolutely continuous**.)

Total Shape Differential

For a **shape-dependent integrand**:

$$J(\Omega) = \int_{\Omega} f(x, \Omega)\, dx,$$

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$$J(\Omega) = \int_\Omega f(x, \Omega)\, dx,$$

where for all fixed $x \in D$:

$$f(x, \Omega_t) = f(x, \Omega_0) + \int_0^t f'(x, \Omega_s)\, ds.$$

Theorem (Total Shape Differential)

Denote $J(t) = J(\Omega_t)$. Then J is absolutely continuous and

$$dJ(\Omega_t; F) = J'(t) = \int_{\Gamma_t} Ff\, d\sigma + \int_{\Omega_t} f'\, dx.$$

A Chain Rule

Scalar, shape-dependent quantity:

$$J(\Omega) = \int_\Omega f(x, G(\Omega))\, dx$$

(Multiple G's are also possible.)

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Theorem (Chain Rule)

Let $f \in C^1$, then $t \mapsto J(t) = J(\Omega_t)$ is absolutely continuous and

$$dJ(\Omega_t; F) = J'(t) = \int_{\Gamma_t} Ff\, d\sigma + \int_{\Omega_t} \frac{\partial f}{\partial G} G'\, dx.$$

Gradients in $\mathbf{H}^1$

Generic form of the shape derivative:

$$dJ(\Omega; F) = \int_\Gamma f(\ldots) F \, d\sigma$$

$\rightarrow$ **linear functional** in $H^1(D)$, operating on F

$dJ(\Omega; \cdot)$ “lives” on **the boundary** Γ (Hadamard-Zolésio structure theorem)

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H^1 Shape Gradient

Riesz representative $F \in H^1(D)$ of $dJ(\Omega; \cdot)$ as **gradient**:

$$\forall G \in H^1(D): \quad \langle F, G \rangle = \int_D (FG + \langle \nabla F, \nabla G \rangle) \, dx = dJ(\Omega; G)$$

→ the solution defines a speed field on **the domain** D

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$$\forall G \in H^1(D): \quad \langle F, G \rangle_\beta = \int_D (FG + \beta \langle \nabla F, \nabla G \rangle) \, dx = dJ(\Omega; G)$$

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Gradient-Descent Method

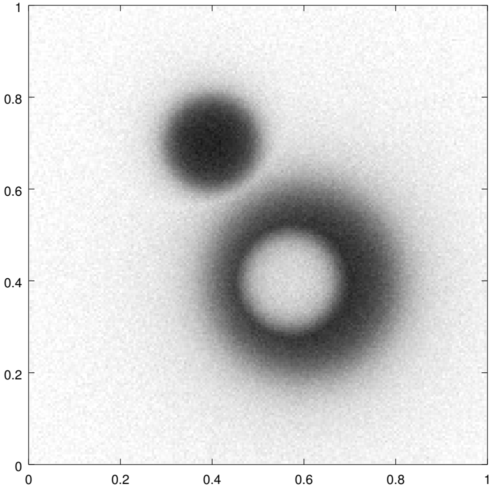
One step in the gradient descent:

1. evaluate **shape-dependent quantities** for Ω
2. calculate **functional** $dJ(\Omega; \cdot)$
3. find Riesz representative: **gradient** F
4. **evolve** Ω with a line search in direction $-F$

Repeat until no more changes are made.

Numerical Results

Image Segmentation



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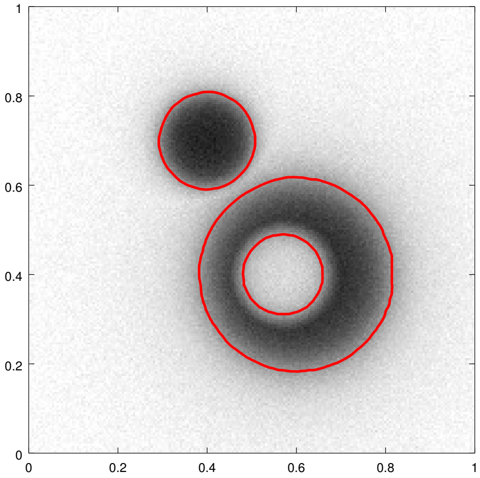

Cost and Shape Derivative

Consider $D \subset \mathbb{R}^n$, e. g., $D = [0, 1]^2$.
Let $u : D \to \mathbb{R}$ be a grey-scale image.

Definition (Cost Function)

$$J(\Omega) = \int_\Omega (u - \overline{u})^2 \, dx - 2\gamma \cdot \sigma \cdot |\Omega| \quad \text{for } \emptyset \neq \Omega \subset D$$

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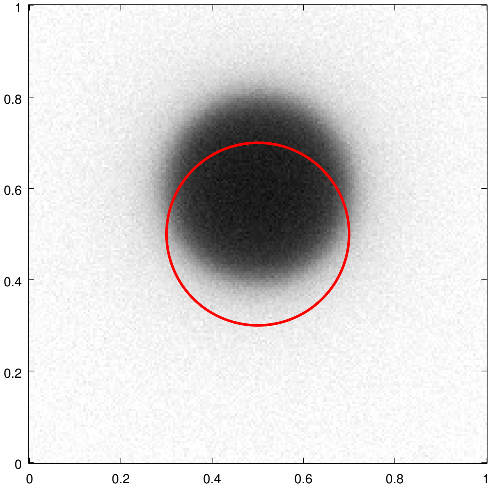
Shape Derivative

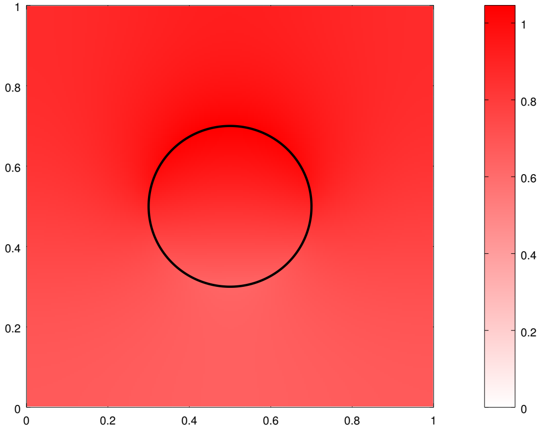
$$dJ(\Omega; F) = \int_\Gamma \left((u - \overline{u})^2 \left(1 - \frac{\gamma}{\sigma} \right) - \gamma\sigma \right) F \, d\sigma$$

Effect of β



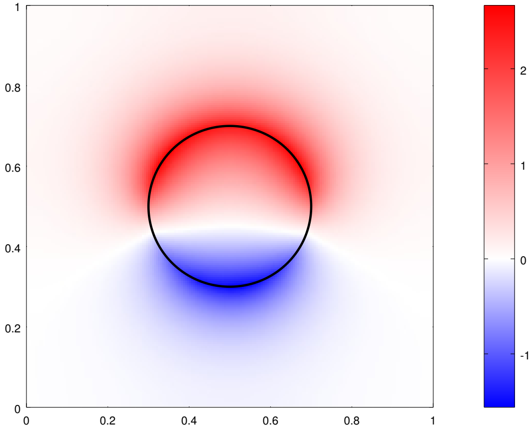
Effect of β

$$\beta = 1$$

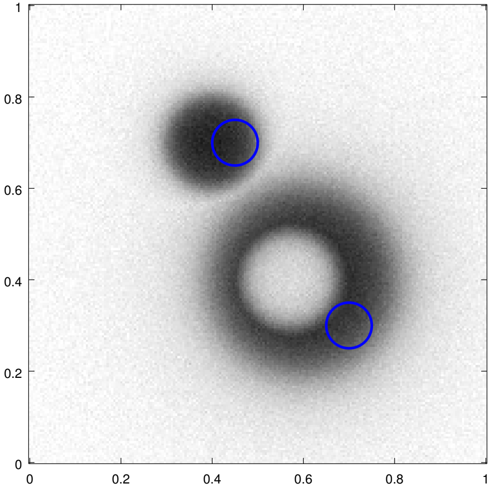


Effect of β

$$\beta = 10^{-2}$$



Descent Run



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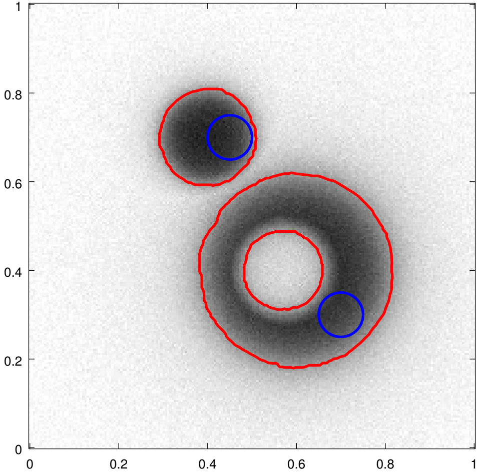

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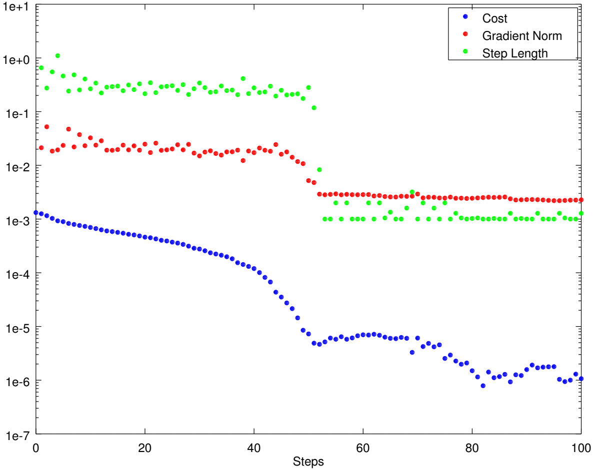

Cost
Gradient Norm
Step Length
1e+1
1e+0
1e-1
1e-2
1e-3
1e-4
1e-5
1e-6
1e-7
0
20
40
60
80
100
Steps

PDE-Constrained Shape Optimisation

Let $D \subset \mathbb{R}^2$ be compact, $B \subset D$, $f \in L^2(D)$ and $u_d \in L^2(B)$.

Definition (Cost Function)

Find Ω with $B \subset \Omega \subset D$ that minimises

$$J(\Omega) = \frac{1}{2} \|u - u_d\|_{L^2(B)}^2 + \alpha |\Gamma| .$$

State Equation

$u \in H^1(\Omega)$ solves the **state equation**:

$$\begin{cases} -\Delta u + u & = & f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} & = & 0 & \text{on } \Gamma \end{cases}$$

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The Shape Derivative

Equations in Weak Form

Find u and p such that (for each $v \in H^1(D)$):

$$\begin{aligned} \int_\Omega (\langle \nabla u, \nabla v \rangle + uv)\, dx &= \int_\Omega fv\, dx \\ \int_\Omega (\langle \nabla p, \nabla v \rangle + pv)\, dx &= \int_B (u - u_d)v\, dx \end{aligned}$$

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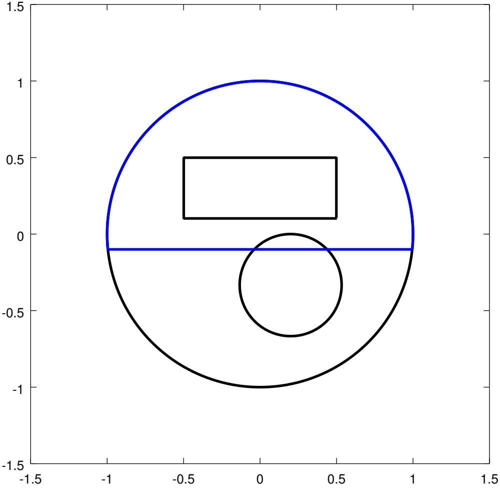
Shape Derivative

$$dJ(\Omega; F) = \int_\Gamma (fp - \langle \nabla u, \nabla p \rangle - up + \alpha\kappa) F \, d\sigma$$

We require: $F = 0$ **on** B

Descent Run



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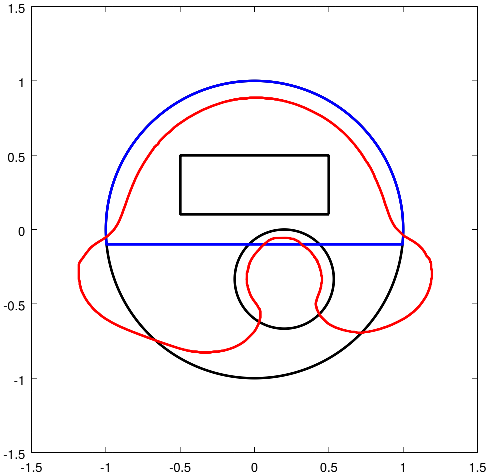

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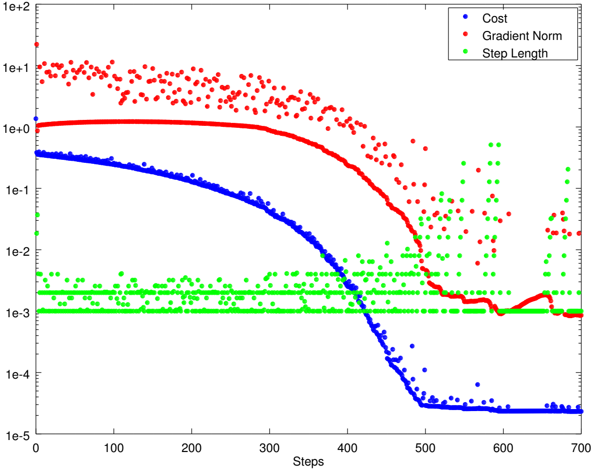
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Conclusion

- Level sets allow a **flexible description of shapes**.
- A **Hopf-Lax formula** can be employed for the time evolution.
- This yields a special **shape calculus**.
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Thanks for your attention!

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