IGDK Munich – Graz

Optimization and Numerical Analysis for Partial Differential Equations with Nonsmooth Structures

A Level-Set Framework for Shape Optimisation

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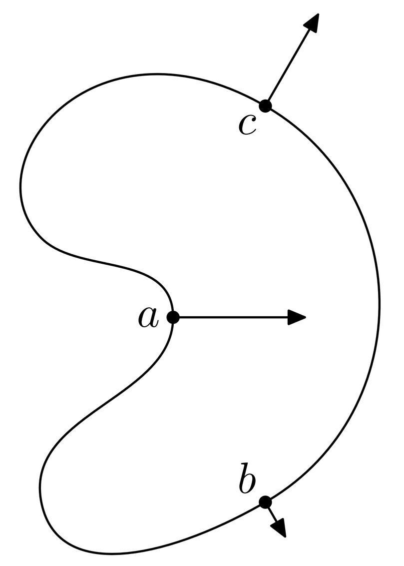
The Level-Set Method

For a level-set function $\phi\colon \mathbb{R}^n \to \mathbb{R}$, we define:

$$\Omega = \phi^{-1}((-\infty, 0)) = \{x \in \mathbb{R}^n \mid \phi(x) < 0\},$$
$$\Gamma = \phi^{-1}(\{0\}) = \{x \in \mathbb{R}^n \mid \phi(x) = 0\}$$

Evolution of $\Omega_0 \subset \mathbb{R}^n$ by the speed method:



Propagation in time with the level-set equation:

$$\phi_t + F(x)\,|\nabla\phi| = 0 \qquad (1)$$

$F \in C^{0,1}(\mathbb{R}^n)$ is a scalar speed field.

Left: $F(a) < 0 < F(b) < F(c)$

State of the Art

- Solution theory for (1) well-established (viscosity solutions); see Giga [10].
- Many successful applications in engineering without rigorous foundation.
- Sethian's Fast Marching Method for efficient numerical computation.
- Some analysis of optimisation in a functional-analytic context; see Burger [9].

Our contribution: A solid measure-theoretic and geometric foundation for the level-set based speed method in shape optimisation.

A Hopf-Lax Representation Formula

The level-set equation (1) can be interpreted as the Hamilton-Jacobi-Bellman equation of a control problem.

If we define the F-induced distance

$$d(x,y) = \inf_{\xi:\, x \rightsquigarrow y} \int_0^1 \frac{|\xi'(\tau)|}{F(\xi(\tau))}\, d\tau, \quad d_0(x) = \inf_{y \in \Omega_0} d(x,y),$$

this gives the time evolution:

$$\phi(x,t) = \inf\{\phi_0(y) \mid d(x,y) \le t\}, \quad \Omega_t = \{x \in \mathbb{R}^n \mid d_0(x) < t\} \qquad (2)$$

Efficient computation of d_0 is possible with the Fast Marching Method to solve the Eikonal equation

$$F(x)\,|\nabla d_0| = 1, \quad d_0 = 0 \text{ on } \overline{\Omega}_0.$$

Shape Sensitivity Analysis

One can use (2) to express a domain functional:

$$J(\Omega_t) = \int_{\Omega_t} f\, dx = J(\Omega_0) + \int_0^t \int_{\Gamma_s} Ff\, d\sigma\, ds$$

This yields a shape calculus that requires little regularity of the involved domains and speed fields.

For shape-dependent integrands like $J(\Omega) = \int_\Omega f(x,\Omega)\, dx$, we get

$$dJ(\Omega; F) = \int_\Gamma Ff\, d\sigma + \int_\Omega f'\, dx.$$

Non-Fattening

Classical result by Barles, Soner, Souganidis [8]:

$$\{(x,t) \mid \phi(x,t) \le 0\} = \overline{\{(x,t) \mid \phi(x,t) < 0\}},$$
$$\{(x,t) \mid \phi(x,t) < 0\} = \{(x,t) \mid \phi(x,t) \le 0\}^\circ$$

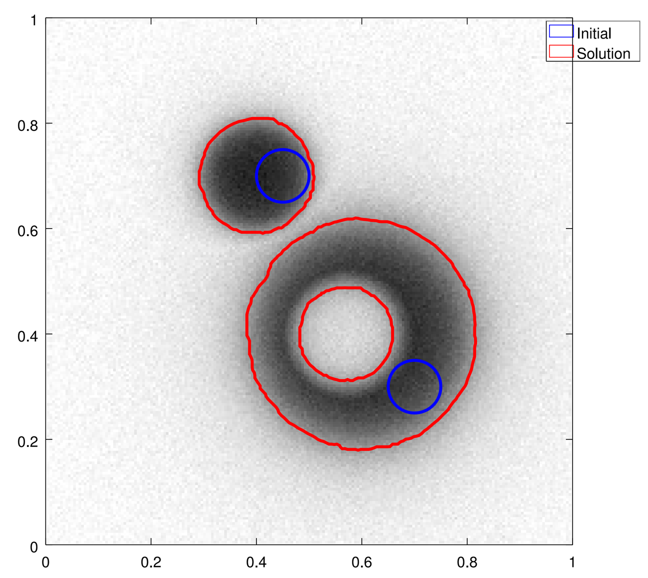
With (2), we can show naturally:

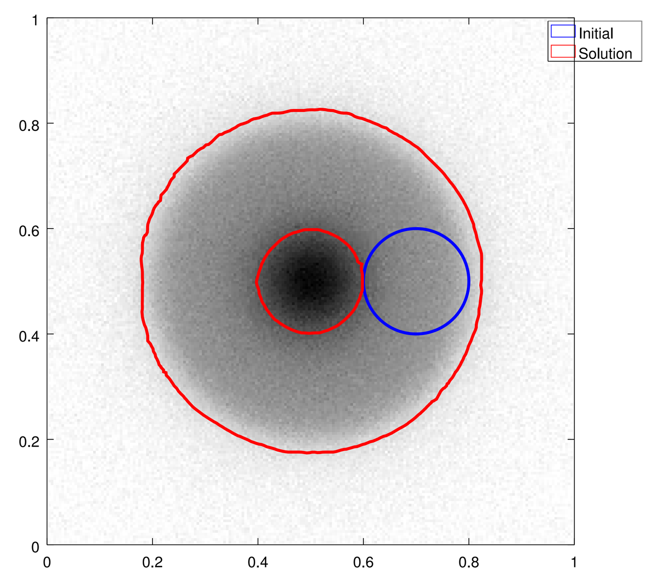
$$F \ge 0 \;\Rightarrow\; \overline{\Omega}_t = \Gamma_t \cup \Omega_t,$$
$$F \le 0 \;\Rightarrow\; (\Gamma_t \cup \Omega_t)^\circ = \Omega_t$$

Image Segmentation

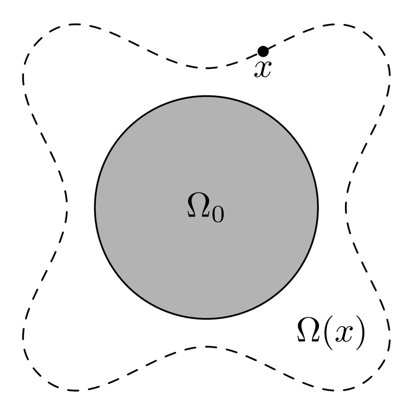
Consider this example problem for image segmentation:

$$J(\Omega) = \int_\Omega (u(x) - \overline{u})^2\, dx - 2\gamma \cdot \sigma \cdot \mathrm{vol}(\Omega)$$





Self-Consistent Gradient Flow



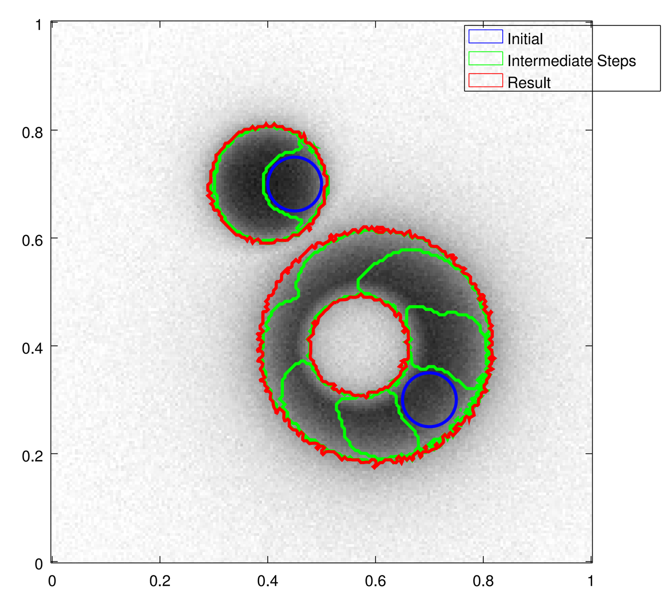
Assume the shape derivative

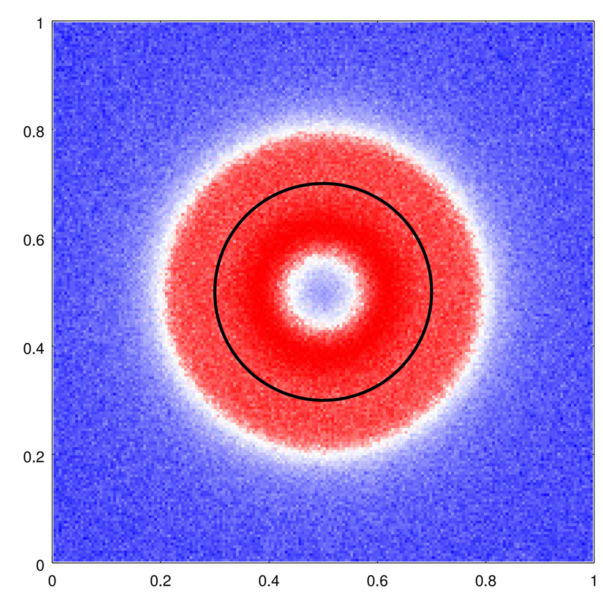
$$dJ(\Omega; F) = \int_\Gamma Ff(x,\Omega)\, d\sigma.$$

We want a self-consistent steepest descent:

$$\psi(F)(x) = F(x) = -f(x, \Omega(x))$$

Under suitable assumptions, existence of a fixed-point speed field can be shown. This gives a self-consistent gradient flow.





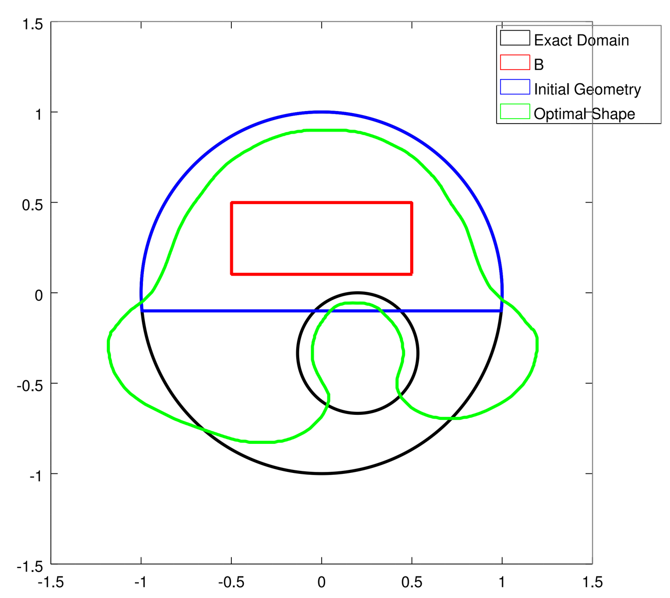
PDE-Constrained Shape Optimisation

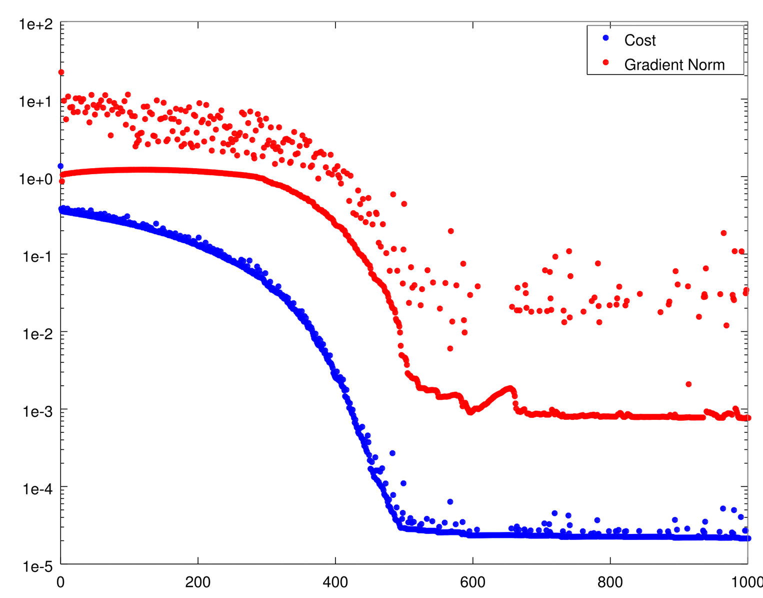
For $f \in L^2(D)$ and $u_d \in L^2(B)$, minimise (with $B \subset \Omega \subset D$):

$$J(\Omega) = \int_B (u - u_d)^2\, dx + R(\Omega),$$
$$-\Delta u + u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega$$

Shape derivative with adjoint approach:

$$dJ(\Omega; F) = \int_\Gamma (fp - \langle \nabla u, \nabla p\rangle - up) F\, d\sigma + dR(\Omega; F)$$





Individual Publications

[1] M. Keuthen and D. Kraft. Shape Optimization of a Breakwater. *Inverse Problems in Science and Engineering (accepted)*, 2015.

[2] D. Kraft. Difficulty Control for Blockchain-Based Consensus Systems. *Peer-to-Peer Networking and Applications*, 2015.

[3] D. Kraft. Measure-Theoretic Properties of Level Sets of Distance Functions. *Journal of Geometric Analysis*

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