Measure-Theoretic Properties of Level Sets of Distance Functions

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Abstract

We consider the level sets of distance functions from the point of view of geometric measure theory. This lays the foundation for further research that can be applied, among other uses, to the derivation of a shape calculus based on the level-set method. Particular focus is put on the $(n-1)$-dimensional Hausdorff measure of these level sets. We show that, starting from a bounded set, all sub-level sets of its distance function have finite perimeter. Furthermore, if a uniform-density condition is satisfied for the initial set, one can even show an upper bound for the perimeter that is uniform for all level sets. Our results are similar to existing results in the literature, with the important distinction that they hold *for all* level sets and not just almost all. We also present an example demonstrating that our results are sharp in the sense that no uniform upper bound can exist if our uniform-density condition is not satisfied. This is even true if the initial set is otherwise very regular (i. e., a bounded Caccioppoli set with smooth boundary).

Keywords: Geometric Measure Theory, Level Set, Distance Function, Hausdorff Measure, Perimeter, Caccioppoli Set

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1 Introduction

Regularity of level sets is a wide and interesting field. This can be seen already in the context of the classical Sard theorem and the co-area formula (see page 112 of [11]). For recent work in this direction, let us refer to [1]. In a more specific context, we are interested in the $(n-1)$-dimensional Hausdorff measure of the level sets of distance functions:

For some open set $\Omega_0 \subset \mathbb{R}^n$, the distance function is given by

$$d_{\Omega_0}(x) = \inf_{y \in \Omega_0} |x - y| .$$

This is a widely studied construct with well-known properties. See, for instance, Chapter 6 of [9]. In particular, note that $d_{\Omega_0}(x)$ is well-defined and non-negative for all $x \in \mathbb{R}^n$. Furthermore, the function d_{Ω_0} is continuous on $\mathbb{R}^n$ and $d_{\Omega_0}(x) = 0$ for all $x \in \overline{\Omega}_0$. For $t > 0$, let us also define the *level sets*

$$\begin{aligned} \Omega_t &= {d_{\Omega_0}}^{-1}\left((-\infty, t)\right) = \{x \in \mathbb{R}^n \mid d_{\Omega_0}(x) < t\} \text{ and} \\ \Gamma_t &= {d_{\Omega_0}}^{-1}\left(\{t\}\right) = \{x \in \mathbb{R}^n \mid d_{\Omega_0}(x) = t\} . \end{aligned} \tag{1}$$

Continuity of the distance function implies that Ω_t is open and Γ_t closed. It is also easy to see that Γ_t coincides with the topological boundary of Ω_t, i. e., $\Gamma_t = \partial\Omega_t$. The set Ω_t is sometimes called the *t-envelope* of Ω_0 in the literature. It is an inflated and smoothed version of Ω_0. The study of the surface measure $\mathcal{H}^{n-1}(\Gamma_t)$, which corresponds roughly to the perimeter $P(\Omega_t)$, is naturally connected to surface flows. This situation was studied by Caraballo in [7] and can be directly motivated by the famous paper [2] of Almgren, Taylor and Wang.

Our main results are very similar, but they differ in one important aspect: They do not depend on the co-area formula. Consequently, they hold *for all* level sets and not just almost all. To highlight a particular use of this improvement, let us briefly mention the classical *level-set method* introduced by Osher and Sethian in [15]: Based on this method, one can describe evolving shapes as the sub-zero level sets of a time-dependent level-set function. If the geometry is changed by moving the boundary in the normal direction according to a given speed field, the time evolution of the level-set function can be described by the so-called *level-set equation*. Based on this method, one can, for instance, build a framework for shape optimisation as done in [5]. See [10], [6] and [14] for some recent applications. Our work in [13] allows us to represent the propagating domains in the level-set framework with a formula similar to (1). In the general case, one uses the solution of an Eikonal equation instead of the distance function d_{Ω_0}. The situation considered here is a special case, which results if the speed field is positive and constant throughout all of $\mathbb{R}^n$.

In this framework, one can also perform shape-sensitivity analysis. The resulting shape derivatives are directly connected to the perimeter of Ω_t and, a-priori, defined only for almost all $t \geq 0$. For the analysis of optimisation methods based on these derivatives, it is important to consider the ability to *continuously extend* the shape derivatives to all times t. Consequently, it is important to ask the question of *continuity of the perimeter* $P(\Omega_t)$ with respect to t. One half of this question can be resolved quite easily by the well-known lower semi-continuity property of the total variation. This, in turn, implies lower semi-continuity for the perimeter of the evolving sets. Estimates in the other direction, however, are more difficult to obtain. In particular, we need upper bounds for $P(\Omega_t)$ that are not restricted by an almost-all qualification in t. While this paper can not give a full continuity result, we are, indeed, able to improve upon the existing results in [7] in this direction. With the developed technical tools, it may be possible to prove continuity of the perimeter in a future work.

In Section 2, we give an example that demonstrates that blow-up of the perimeter of Ω_t can happen for $t \to 0^+$ even if Ω_0 is a smooth Caccioppoli set. After investigating some auxiliary geometric properties of spherical sectors in Section 3, we will derive our main results in Section 4. The first is a kind of inverse isoperimetric inequality (see Theorem 3), that gives an upper bound on the perimeter $P(\Omega_t)$ of Ω_t in terms of the created volume $\Omega_t \setminus \Omega_0$. An obvious estimate of this volume follows if Ω_0 is bounded, which results in Corollary 1. Note, however, that this only yields an upper bound for $P(\Omega_t)$ that diverges like $1/t$ for $t \to 0^+$. This matches our observations in Theorem 1. Under an additional uniform-density assumption on Ω_0, we can further improve the estimate: In this situation, the volume can be bounded in terms of the perimeter of the initial domain Ω_0 times t. Consequently, we obtain a *uniform* bound on the perimeter of Ω_t. This will be done in Subsection 4.2. Subsection 4.3 discusses our uniform-density condition in comparison to related geometric properties in the literature. We will see that it is strictly weaker than the uniform cone property, and a sufficient condition for the finite density perimeter introduced by Bucur and Zolésio in [4]. Note that our main results and the uniform-density assumption are sharp, as demonstrated by the counterexamples in Section 2.

2 Motivating Example for Perimeter Blow-Up

Before we start working towards the main results, let us give a motivating example. It shows why it is necessary to introduce the notion of uniform lower density in Subsection 4.2 together with the complexities it creates.

There is a classical textbook example for elementary geometry: Let a rope be put tightly around the Earth's equator. If the rope is now prolonged by a single metre, how far will it be above the surface? With a trivial calculation, one arrives at the surprising result that the distance is not negligible. In fact, the relationship between the changes in a circle's radius and its perimeter is *independent of the circle's size*. We can exploit this fact not just for huge but also for tiny circles. This allows us to show that the perimeter of Ω_t can blow up for $t \to 0^+$ even if Ω_0 has finite perimeter and is bounded:

Example 1. Consider $D = [0,2] \times [0,1] \subset \mathbb{R}^2$ as hold-all domain. For $k = 0, 1, \ldots$, define

$$l_k = 4^{-k}, \quad r_k = \frac{(l_k)^2}{4} = \frac{1}{4} \cdot 16^{-k}, \quad N_k = \frac{2^{-k}}{(l_k)^2} = 8^k.$$

Based on these definitions, we define Ω_0 as an infinite union of balls as depicted in Figure 1. Specifically, Ω_0 is constructed by splitting D first into a sequence of vertical strips with widths 2^{-k}. Each strip is then further divided into squares of size $l_k \times l_k$. Into each such square, we put a ball with radius r_k. For each k, there is a total of N_k such squares and balls.

Each ball at level k has perimeter $2\pi r_k$, so that the total perimeter of Ω_0 is given as

$$P(\Omega_0) = \sum_{k=0}^{\infty} N_k \cdot 2\pi r_k = \frac{\pi}{2} \sum_{k=0}^{\infty} \left(\frac{8}{16}\right)^k = \pi.$$

Thus, Ω_0 is a bounded set of finite perimeter. It is also clear that it has a smooth boundary, since it consists entirely of balls. However, since the radii of the balls become arbitrarily small, the curvature of Γ_0 is not bounded.

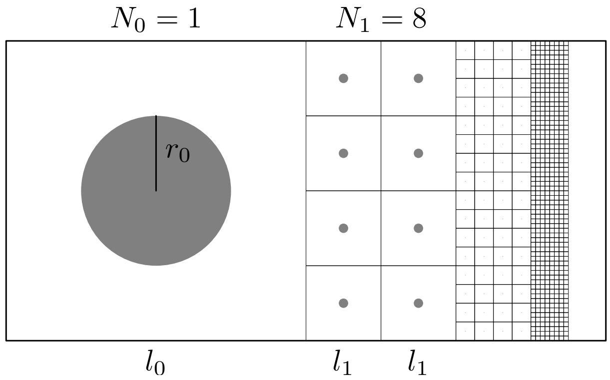


Figure 1: The notation and initial set Ω_0 used in Example 1. Ω_0 consists of the union of all grey balls.

For the time evolution of Ω_0, note that each circle grows outwards and is a circle of radius $r_k + t$ at time t. This works as long as t is small enough, so that the circle does not yet hit another growing circle. If we let t_k be the time at which the circles of level k hit their enclosing square, we find that

$$t_k = \frac{l_k}{2} - r_k = \frac{1}{2} \cdot 4^{-k} - \frac{1}{4} \cdot 16^{-k} = \frac{1}{2} \cdot 4^{-k} \left(1 - \frac{1}{2} \cdot 4^{-k}\right) \geq \frac{1}{4} \cdot 4^{-k}. \tag{2}$$

The other way round, this means that for times $t < t_k$, all circles up to (and including) level k have certainly not touched any others. Let $t > 0$ be given, and m such that $t_{m+1} \leq t < t_m$. If we use only circles up to level m to estimate the perimeter of Ω_t, this yields

$$P(\Omega_t) \geq \sum_{k=0}^{m} N_k \cdot 2\pi(r_k + t) \geq 2\pi \sum_{k=0}^{m} N_k t \geq 2\pi t_{m+1} \sum_{k=0}^{m} 8^k \geq \frac{\pi}{2} \frac{1}{4^{m+1}} \frac{8^{m+1} - 1}{7} \geq \frac{\pi}{14} \left(2^{m+1} - 1\right). \tag{3}$$

Note that we can see already here that this expression is unbounded for $t \to 0^+$, since this limit corresponds to $m \to \infty$. To get a more precise estimate, we can rewrite (2) to get

$$4^m \geq \frac{1}{4t_m} \quad \Leftrightarrow \quad 2^m \geq \frac{1}{2\sqrt{t_m}} \quad \Rightarrow \quad 2^{m+1} \geq \frac{1}{2\sqrt{t_{m+1}}} \geq \frac{1}{2\sqrt{t}}.$$

Combining this result with (3) finally gives

$$P(\Omega_t) \geq \frac{\pi}{14} \left(\frac{1}{2\sqrt{t}} - 1\right),$$

which diverges like $1/\sqrt{t}$ as $t \to 0^+$ and certainly becomes unbounded.

If one considers the calculations in Example 1 carefully, one can see that the base number in the definition of l_k (four in the example) influences only the constant in front of the final estimate as long as it is larger than two. The exponent $1/2$ determining the rate to be $1/\sqrt{t}$ comes from the fact that each level of balls gets assigned only *half* the area that was assigned to the previous level. We can increase this fraction as long as it is *less than one* if we still want to get a bounded set as result. This line of thought can be extended to the following result:

Theorem 1. *Let $n \geq 2$ and $0 < s < 1$ be given. There exists a Caccioppoli set $\Omega_0 \subset \mathbb{R}^n$ bounded and with smooth boundary, such that*

$$P(\Omega_t) \geq \frac{C}{t^s}$$

for some constant C and $t > 0$ small enough. In particular, this rate of divergence holds in the limit $t \to 0^+$.

Proof. We replicate the construction of Example 1: For the desired result, choose some $\alpha > 1$ and set

$$f = \alpha^{s-1} \in (0, 1).$$

Note that $f\alpha^n > f\alpha = \alpha^s > 1$. We define

$$l_k = \alpha^{-k}, \quad r_k = \frac{(l_k)^n}{4} = \frac{\alpha^{-kn}}{4}, \quad N_k = \left\lceil f^k \alpha^{nk} \right\rceil.$$

This leads to a total volume of all $(l_k)^n$-cubes of

$$\sum_{k=0}^{\infty} N_k(l_k)^n \leq \sum_{k=0}^{\infty} \left(f^k \alpha^{nk} + 1\right) \alpha^{-nk} = \sum_{k=0}^{\infty} f^k + \sum_{k=0}^{\infty} \left(\alpha^{-n}\right)^k = \frac{1}{1-f} + \frac{1}{1-\alpha^{-n}} < \infty.$$

Hence, since $f < 1$, we can fit everything into a bounded set as before. Clearly, Ω_0 has again a smooth boundary. Its perimeter is also finite since

$$\begin{aligned} P(\Omega_0) &= C \sum_{k=0}^{\infty} N_k (r_k)^{n-1} \leq C \sum_{k=0}^{\infty} \left(f^k \alpha^{nk} + 1\right) r_k \\ &= \frac{C}{4} \left(\sum_{k=0}^{\infty} f^k + \sum_{k=0}^{\infty} \left(\alpha^{-n}\right)^k \right) = \frac{C}{4} \left(\frac{1}{1-f} + \frac{1}{1-\alpha^{-n}} \right). \end{aligned}$$

On the other hand, we still find that balls at level k have not yet hit anything else until time

$$t_k = \frac{l_k}{2} - r_k = \frac{\alpha^{-k}}{2} - \frac{\alpha^{-kn}}{4} \geq \frac{\alpha^{-k}}{2} - \frac{\alpha^{-k}}{4} = \frac{\alpha^{-k}}{4}. \tag{4}$$

Thus, for $t > 0$ with $t_{m+1} \leq t < t_m$, we know that

$$\begin{aligned} P(\Omega_t) &\geq C \sum_{k=0}^{m} N_k (r_k + t)^{n-1} \geq C (t_{m+1})^{n-1} \cdot \sum_{k=0}^{m} N_k \geq C \left(\frac{\alpha^{-(m+1)}}{4} \right)^{n-1} \cdot \sum_{k=0}^{m} (f \alpha^n)^k \\ &= \frac{C}{4^{n-1}} \left(\alpha^{n-1}\right)^{-(m+1)} \frac{f^{m+1} (\alpha^n)^{m+1} - 1}{f \alpha^n - 1} \geq \frac{C}{4^{n-1} (f\alpha^n - 1)} \left((f\alpha)^{m+1} - 1 \right) = C' \left((\alpha^s)^{m+1} - 1 \right), \end{aligned}$$

where we have defined the constant C' suitably. From (4), it follows that

$$\alpha^{m+1} \geq \frac{1}{4 t_{m+1}} \geq \frac{1}{4t} \quad \Leftrightarrow \quad (\alpha^s)^{m+1} \geq \frac{4^{-s}}{t^s}.$$

Combining this with the estimate for $P(\Omega_t)$ above shows the claim. □

3 Auxiliary Geometric Results

In order to show our main results in Section 4 (in particular, Theorem 3), we need some auxiliary results. They are only based on elementary geometry and will be prepared in this section. The basic object studied is what we will call a *sector* below:

Definition 1. Let $x_0, x \in \mathbb{R}^n$ and $\phi \in [0, \pi/2]$. We define

$$S_\phi(x_0, x) = \{ y \in \mathbb{R}^n \mid 0 < |x_0 - y| < |x_0 - x| \text{ and } (y - x_0) \cdot (x - x_0) > |x_0 - y| \, |x_0 - x| \cdot \cos \phi \}.$$

We will often set $t = |x_0 - x|$ to be the sector's radius. The set $S_\phi(x_0, x)$ is an open sector of the ball with centre x_0 and radius t. The value of ϕ, which corresponds to the maximum allowed angle x–x_0–y, defines the sector's aperture.

Besides using the angle ϕ directly, we will also need to define such a sector via an auxiliary ball $B_\delta(x)$ for $\delta < t$. The idea is depicted in Figure 2b: In this case, the sector's aperture is defined *indirectly* via δ. It is chosen as the angle at which the ball around x intersects the larger sphere with centre x_0. With basic trigonometry, one can derive

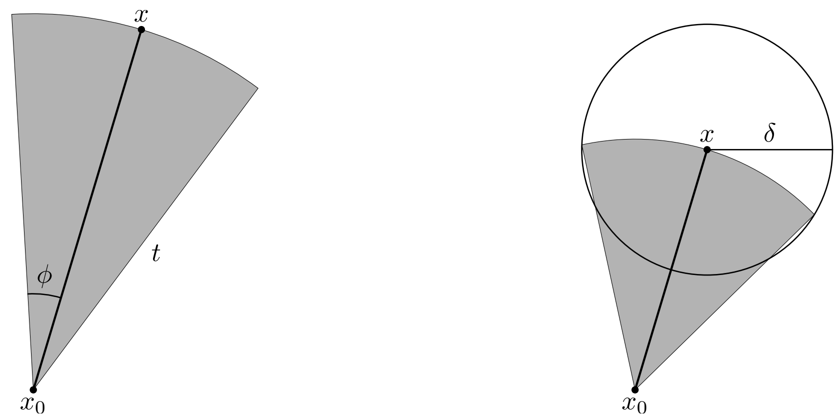
$$\phi(\delta) = \arccos \left(1 - \frac{\delta^2}{2t^2} \right) \tag{5}$$

for the corresponding aperture angle. In the following, we will only need two basic properties of this explicit function: $\delta < t\phi(\delta)$ holds for all δ and $t\phi(\delta)/\delta \to 1$ in the limit $\delta \to 0^+$. In other words, $t\phi(\delta) \approx \delta$ asymptotically for small δ.

The first part of our geometric analysis of sectors is concerned with determining their volume (i. e., n-dimensional Lebesgue measure). For this, let us state the following fundamental geometric facts:

Lemma 1. *Let $n \geq 2$. The volume of a ball with radius $\rho > 0$ is given by*

$$\operatorname{vol}(B_\rho(x)) = \omega_n \rho^n, \quad \omega_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}.$$



(a) Based on ϕ according to Definition 1.

(b) Defined via the ball $B_\delta(x)$ and (5).

Figure 2: Definitions of the sector $S_\phi(x_0, x)$.

This holds obviously for arbitrary $x \in \mathbb{R}^n$.

Furthermore, there exists a mapping $r\colon [0, \pi/2] \to [0, 1/2]$ which is continuous, bijective, strictly increasing and satisfies

$$\operatorname{vol}(S_\phi(x_0, x)) = r(\phi) \cdot \operatorname{vol}(B_t(0)) = r(\phi) \cdot \omega_n t^n \tag{6}$$

for all $x_0, x \in \mathbb{R}^n$ and $\phi \in [0, \pi/2]$. Here, we have set $t = |x_0 - x|$ as before. In addition,

$$\lim_{\phi \to 0^+} \frac{r(\phi)}{\phi^{n-1}} > 0 \tag{7}$$

exists and is strictly positive.

Proof. The volume of n-dimensional balls is a well-known result. See, for instance, Theorem 26.13 in [16]. The remaining statements follow by a routine calculation in spherical coordinates. □

We can also relate the surface area of a sector's base to its volume. This result will be used later when we prove Theorem 3. It follows immediately from Lemma 1 and, in particular, (7):

Lemma 2. *For fixed $t > 0$, there exist $\delta_0 > 0$ and a dimensional constant C such that*

$$\delta^{n-1}\omega_{n-1} \le C \frac{\operatorname{vol}\left(S_{\phi(\delta)}(x_0, x)\right)}{t}$$

for all $\delta \in (0, \delta_0)$ and arbitrary $x_0, x \in \mathbb{R}^n$ with $|x_0 - x| = t$.

Finally, let us consider *two* sectors $S_\phi(x_0, x)$ and $S_\phi(y_0, y)$. The angle is the same for both, and we assume that $\phi = \phi(\delta)$ for some $\delta > 0$. Let also $t = |x_0 - x| = |y_0 - y|$. We are particularly interested in the situation

$$\overline{B_\delta(x)} \cap \overline{B_\delta(y)} = \emptyset \;\text{ and }\; t \le \min(|x_0 - y|, |y_0 - x|). \tag{8}$$

A main ingredient for the proof of Theorem 3 is the fact that this condition is sufficient for both sectors to be disjoint. This is illustrated in Figure 3: If the balls are disjoint, we can construct the hyperplane H that divides the line x–y at its midpoint and is perpendicular to it. This plane has the property that all points "above" it are closer to x than to y, and vice-versa for points on the other side. Thus, (8) implies that x_0 is on the same side as x, while y_0 must be on the other side together with y. Hence, the plane separates the convex sets $S_\phi(x_0, x)$ and $S_\phi(y_0, y)$ from each other, which means that the sectors must be disjoint. This is the main idea behind the following result:

Lemma 3. *Let $\delta > 0$ and $x_0, y_0, x, y \in \mathbb{R}^n$ with $t = |x_0 - x| = |y_0 - y|$ such that (8) holds. Then*

$$S_{\phi(\delta)}(x_0, x) \cap S_{\phi(\delta)}(y_0, y) = \emptyset.$$

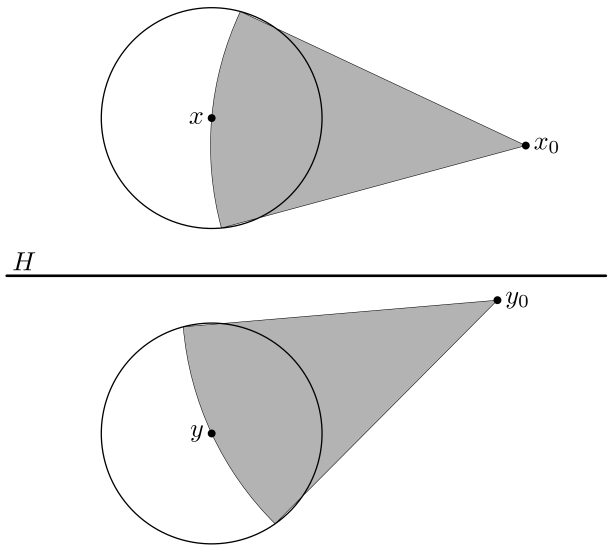


Figure 3: The hyperplane H separates both the balls and the full sectors from each other when (8) holds. This is the main idea in the proof of Lemma 3.

Proof. With a proper translation, we can assume, without loss of generality, that $y = -x$. Because $\overline{B_\delta\left(x\right)}$ and $\overline{B_\delta\left(y\right)}$ are disjoint, the hyperplane

$$H = \{p \in \mathbb{R}^n \mid x \cdot p = 0\}$$

separates both balls (see Figure 3). Furthermore, by (8) we know

$$|x_0 - x|^2 = t^2 \leq |x_0 - y|^2 = |x_0 + x|^2 .$$

Multiplying this inequality out, we find $0 \leq x \cdot x_0$. This means that x and x_0 are on the same side of H. Similarly, we also find that y and y_0 are on one side of H. Since $y = -x$, this means

$$0 \leq y \cdot y_0 \quad \Leftrightarrow \quad x \cdot y_0 \leq 0.$$

Hence, x_0 and y_0 are *on different sides* of the hyperplane H. Thus, H separates also the convex hulls of $\overline{B_\delta\left(x\right)} \cup \{x_0\}$ and $\overline{B_\delta\left(y\right)} \cup \{y_0\}$, which contain $S_\phi(x_0, x)$ and $S_\phi(y_0, y)$, respectively. This shows that the sectors are, indeed, disjoint. □

4 Main Results

With the preparations of Section 3 in place, we can now proceed to show the main results. As before, let us assume that $\Omega_0 \subset \mathbb{R}^n$ is an open set. We denote its boundary by $\Gamma_0 = \partial\Omega_0$ and introduce $d = d_{\Omega_0}$ as the distance function of Ω_0. Recall also the definitions of Ω_t and Γ_t from (1).

Lemma 4. *For each $x \in \mathbb{R}^n \setminus \Omega_0$,*

$$d(x) = \inf_{y \in \Gamma_0} |x - y| . \tag{9}$$

Furthermore, there exists $x_0 \in \Gamma_0$ with $d(x) = |x - x_0|$.

Proof. See (2.2) on page 337 of [9] for (9). Γ_0 is closed, and we can clearly restrict the infimum to some bounded subset of Γ_0. Hence, this subset is compact and there exists a minimiser x_0. □

In the following, we are interested in estimating the "surface area" of Ω_t for $t > 0$. Before we can do that, let us briefly recall the applicable concepts for defining such a surface area in the first place: For an open set $\Omega \subset \mathbb{R}^n$, we denote by $P\left(\Omega\right)$ its *perimeter* as defined, for instance, by Definition 3.35 on page 143 of [3]. (Note that we are mostly interested in the perimeter relative to the base set $\mathbb{R}^n$.) The set Ω is said to have finite perimeter or to be a *Caccioppoli set* if $P\left(\Omega\right) < \infty$.

Furthermore, let us introduce also the *Hausdorff measure* following Definition 2.46 on page 72 of [3]:

Definition 2. Let $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$. For $\delta > 0$, we define

$$\mathcal{H}^k_\delta(\Omega) = \inf \left\{ \sum_{i=1}^{\infty} \left(\frac{d_i}{2}\right)^k \omega_k \;\middle|\; \Omega \subset \bigcup_{i=1}^{\infty} U_i,\ d_i = \sup_{x,y \in U_i} |x-y|\,,\ d_i \le 2\delta \right\}.$$

Here, ω_k denotes the volume of the k-dimensional unit ball as in Lemma 1. The value d_i is the *diameter* of the set U_i, and it is allowed to be at most 2δ in order for (U_i) to be an admissible δ-covering of Ω.

Furthermore, the *k-dimensional Hausdorff measure* of Ω is then given by

$$\mathcal{H}^k(\Omega) = \sup_{\delta > 0} \mathcal{H}^k_\delta(\Omega) = \lim_{\delta \to 0+} \mathcal{H}^k_\delta(\Omega).$$

Note that we define the Hausdorff measure in such a way that $\mathcal{H}^n$ corresponds to the n-dimensional Lebesgue measure. (For a proof, see Theorem 2.53 in [3].) This is the reason for including ω_k in the definition. Other authors (e. g., [16]) do not add this normalisation constant, which results in a notion of $\mathcal{H}^k$ that is different from Definition 2 by a constant.

For the case of only one dimension, the situation is simple since sets of finite perimeter in one dimension can be represented (up to a set of measure zero) as the union of a finite number of intervals:

Theorem 2. *Let $n = 1$ and $\Omega_0 \subset \mathbb{R}$ be open and bounded. Then Γ_t is a finite set for each $t > 0$ and its cardinality is non-increasing with respect to t. Furthermore,*

$$\mathcal{H}^0(\Gamma_t) \le P(\Omega_0)\,. \tag{10}$$

If Ω_0 has finite perimeter and t is sufficiently small, then both values are actually equal.

Proof. Let $t > 0$ and $x \in \Gamma_t$. Lemma 4 implies that there exists $x_0 \in \Gamma_0$ with $|x - x_0| = t$. Assume, without loss of generality, that $x_0 < x$. It follows that $I_x = (x_0, x) \subset d^{-1}((0,t))$. Furthermore, if $y \in \Gamma_t$ and $x \ne y$, then $I_x \cap I_y = \emptyset$. Since $\operatorname{vol}(I_x) = t > 0$ for each $x \in \Gamma_t$ and Ω_t is bounded, the cardinality of Γ_t is bounded as $\mathcal{H}^0(\Gamma_t) \le \operatorname{vol}(\Omega_t)/t$ and thus finite. If we have $0 < s < t$, the estimate (10) implies that

$$\mathcal{H}^0(\Gamma_t) \le P(\Omega_s) \le \mathcal{H}^0(\Gamma_s).$$

Hence it follows that the cardinality is non-increasing when we have established (10).

For (10), assume that Ω_0 has finite perimeter (the situation is trivial otherwise). According to Proposition 3.52 on page 153 of [3], there exist $p \in \mathbb{N}$ and p disjoint intervals $J_i = [a_i, b_i]$ such that $\Omega_0 \subset \bigcup_{i=1}^p J_i$. These two sets can only differ up to a set of measure zero. Furthermore, $P(\Omega_0) = 2p$. As before, we can associate an interval $I_x \subset d^{-1}((0,t))$ to each $x \in \Gamma_t$, and all I_x are disjoint. If we assume that $I_x = (x_0, x)$, then $x_0 = b_i$ for some $1 \le i \le p$. Similarly, $x_0 = a_i$ if $I_x = (x, x_0)$. This implies (10), since

$$\mathcal{H}^0(\Gamma_t) \le 2p = P(\Omega_0)\,.$$

If we assume an ordering such as

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_p < b_p$$

and denote by

$$L = \inf_{i=1,\ldots,p-1} (a_{i+1} - b_i) > 0$$

the minimal distance between the intervals J_i, then equality holds with $\mathcal{H}^0(\Gamma_t) = 2p$ for $t < L/2$. □

4.1 A Bound on the Hausdorff Measure

Intuitively, Ω_t is constructed from Ω_0 by adding a "layer" of thickness t onto Γ_0. Following this picture, one can imagine that the volume of this layer should roughly equal t times the surface area (i. e., perimeter) of either Ω_0 or Ω_t. This argument can be made rigorous by estimating the volume in terms of $P(\Omega_0)$, and then $\mathcal{H}^{n-1}(\Gamma_t)$ in terms of the volume. The former will be done in Subsection 4.2. We will show the latter as our first main result in this subsection. This is, somehow, an *inverse isoperimetric inequality.* Of course, in the general situation no inverse to the classical isoperimetric inequality (see Subsection 5.6.2 of [11]) holds. In our case, however, it works because the considered volume is not allowed to be "arbitrarily thin".

Definition 3. For a fixed initial set Ω_0 and $t > 0$, we define the *newly created volume* to be

$$U_t = \left(\bigcup_{x_0 \in \Gamma_0} B_t(x_0) \right) \setminus \overline{\Omega}_0 = \{ x \in \mathbb{R}^n \mid 0 < d(x) < t \}\,.$$

We can now state and prove the first main result:

Theorem 3. *There exists a dimensional constant C such that*

$$P(\Omega_t) \le \mathcal{H}^{n-1}(\Gamma_t) \le C \cdot \frac{\operatorname{vol}(U_t)}{t}$$

holds for all $t > 0$.

Proof. The first inequality is a well-known fact about the relation between perimeter and the Hausdorff measure. See, for instance, Proposition 3.62 on page 159 of [3]. We will now show the second inequality. For this, let $\delta > 0$ be given. Then clearly $\Gamma_t \subset \cup_{x\in\Gamma_t} \overline{B_\delta(x)}$. According to Vitali's covering theorem (see Theorem 1 on page 27 of [11]), there exists a countable subset $X \subset \Gamma_t$ such that

$$\Gamma_t \subset \bigcup_{x\in X} \overline{B_{5\delta}(x)} \tag{11}$$

and all $\overline{B_\delta(x)}$ are disjoint for $x \in X$. Note that X is, in fact, finite if Ω_0 and thus also Γ_t are bounded.

For each $x \in \Gamma_t$, there exists a corresponding $x_0 \in \Gamma_0$ with $|x - x_0| = t$ according to Lemma 4. Furthermore, $t \le |y - y_0|$ for all $y \in \Gamma_t$ and $y_0 \in \Gamma_0$. For $x \in X$ and its associated point $x_0 \in \Gamma_0$, let us define

$$S_x = S_{\phi(\delta)}(x_0, x).$$

Note that the condition (8) is satisfied for each pair (S_x, S_y) with $x, y \in X$, so that all S_x and S_y with $x \ne y$ are disjoint by Lemma 3. Also note that a basic geometric argument implies $S_x \cap \overline{\Omega}_0 = \emptyset$ for small enough δ. Thus, we find that each S_x is contained in the newly created volume and get

$$\sum_{x\in X} \operatorname{vol}(S_x) = \operatorname{vol}\left(\bigcup_{x\in X} S_x\right) \le \operatorname{vol}(U_t). \tag{12}$$

Since the enlarged balls in (11) provide a particular 5δ-covering of Γ_t, we know that

$$\mathcal{H}^{n-1}_{5\delta}(\Gamma_t) \le \sum_{x\in X} (5\delta)^{n-1}\omega_{n-1} \le 5^{n-1}\frac{C'}{t}\sum_{x\in X}\operatorname{vol}(S_x).$$

The last estimate and the constant C' come from Lemma 2. Together with (12), this yields

$$\mathcal{H}^{n-1}_{5\delta}(\Gamma_t) \le 5^{n-1}C' \cdot \frac{\operatorname{vol}(U_t)}{t}.$$

The bound on the right-hand side does not depend on δ any more, so that we can take the limit $\delta \to 0^+$ to finish the proof. □

Having this first result, we can already show that all evolved sets Ω_t must be Caccioppoli sets:

Corollary 1. *Let Ω_0 be bounded. Then Ω_t has finite perimeter for all $t > 0$.*

Proof. From the boundedness of Ω_0, we can directly conclude that also Ω_t and U_t are bounded sets for any fixed t. Thus, $\operatorname{vol}(U_t) < \infty$ and Theorem 3 implies that $\mathcal{H}^{n-1}(\Gamma_t)$ is finite for each t. It follows now again from Proposition 3.62 on page 159 of [3] that Ω_t is a set of finite perimeter. □

Take note that the actual bound we get from Corollary 1 diverges like $1/t$ for $t \to 0^+$. It will be the focus of the next subsection (in particular, Corollary 2) to show a *uniform* bound as $t \to 0^+$ under additional assumptions. Without these assumptions, however, we can not hope for any strong improvement of Corollary 1: As we have seen in Theorem 1, the optimal upper bound must diverge stronger than $1/t^s$ for any $s \in (0, 1)$.

4.2 Uniform Bounds

As we have seen above in Theorem 3, the quantity $\operatorname{vol}(U_t)/t$ is crucial as it gives an upper bound on the evolved sets' perimeters. Particularly interesting is the limit $t \to 0^+$. As our second main result below, we can show that there exists a uniform upper bound for $t \to 0^+$ as long as a *uniform density condition* holds for the initial set Ω_0. This condition prevents arbitrarily sharp corners and cusps. To be precise:

Definition 4. Let $A \subset \Gamma_0$, $c \in (0,1)$ and $t_0 > 0$. We say that Ω_0 has (t_0, c)-*uniform lower density* on A if the estimate

$$0 < c \leq \frac{\mathrm{vol}\left(B_t\left(x\right) \cap \Omega_0\right)}{\mathrm{vol}\left(B_t\left(x\right)\right)} \tag{13}$$

holds for all $t \in (0, t_0)$ and $x \in A$. Similarly, Ω_0 is said to have (t_0, c)-*uniform upper density* on A if

$$\frac{\mathrm{vol}\left(B_t\left(x\right) \cap \Omega_0\right)}{\mathrm{vol}\left(B_t\left(x\right)\right)} \leq 1 - c < 1. \tag{14}$$

When both conditions are satisfied together, Ω_0 simply has (t_0, c)-*uniform density* on A.

For fixed x and in the limit $t \to 0^+$, the quotient in (13) and (14) gives the *density* of Ω_0 at x. See page 158 of [3] for some known results about this quantity. In particular, let $\mathcal{F}\Omega$ denote the *reduced boundary* of an open set Ω. (Roughly speaking, this is the set of all boundary points where a measure-theoretic variant of the normal vector to the boundary can be defined. See Definition 3.54 on page 154 of [3].) Then Ω has density $1/2$ at all points in $\mathcal{F}\Omega$. This is, for instance, also true in the example constructed in Section 2. Hence, note that *uniformity* of the estimates is really crucial for our purposes in the following. Note that we are not the first to introduce the concept of uniform lower density. It has been used already by others in a similar context. See, for instance, Proposition 4.2 in [2] and Theorem 6 in [7]. The relation between uniform density and other, more established geometric properties will be discussed in more detail in Subsection 4.3.

For our estimate of $\mathrm{vol}\left(U_t\right)$, we need to somehow get an upper bound on t in terms of the perimeter of Ω_0. For a classical result in this direction, see (3.54) on page 156 of [3]. Unfortunately, this estimate is local in nature and not uniform over the whole boundary of Ω_0. Note, however, that (13) and (14) together are equivalent to

$$c \leq \frac{\min\left(\mathrm{vol}\left(B_t\left(x\right) \cap \Omega_0\right), \mathrm{vol}\left(B_t\left(x\right) \setminus \Omega_0\right)\right)}{\mathrm{vol}\left(B_t\left(x\right)\right)}. \tag{15}$$

This relation can be combined with the *relative isoperimetric inequality* (see, for instance, Theorem 2 on page 190 of [11]) to get the uniform estimate that we need:

Lemma 5. *Let Ω_0 have (t_0, c)-uniform density on A. Then there exists a dimensional constant C such that*

$$t^{n-1} \leq C \left(\frac{1}{c}\right)^{\frac{n-1}{n}} \mathcal{H}^{n-1}\left(B_t\left(x\right) \cap \mathcal{F}\Omega_0\right)$$

for all $x \in A$ and $t \in (0, t_0)$.

Proof. Since we assume uniform density, (15) implies that

$$c \cdot \mathrm{vol}\left(B_t\left(x\right)\right) = c \cdot \omega_n t^n \leq \min\left(\mathrm{vol}\left(B_t\left(x\right) \cap \Omega_0\right), \mathrm{vol}\left(B_t\left(x\right) \setminus \Omega_0\right)\right)$$

for all $t \in (0, t_0)$. If we also apply the relative isoperimetric inequality, we get

$$t^{n-1} \leq C \left(\frac{1}{c}\right)^{\frac{n-1}{n}} P\left(\Omega_0; B_t\left(x\right)\right)$$

for some dimensional constant C. This implies the result together with the well-known relation between perimeter and $\mathcal{H}^{n-1}$ that can be found in Theorem 3.59 on page 157 of [3]. ☐

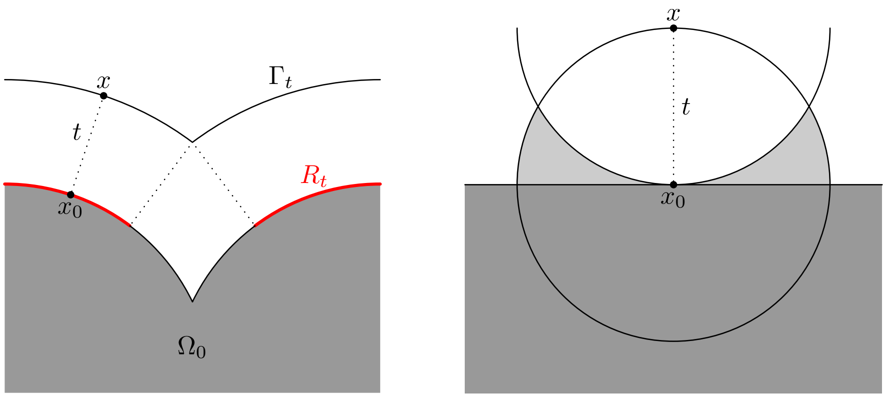
So far, we have assumed uniform density of Ω_0. It will turn out, however, that it is enough to require only uniform *lower* density. Uniform upper density is provided automatically if we choose the subset $A \subset \Gamma_0$ in the right way:

Definition 5. We say that $x_0 \in \Gamma_0$ is *backwards reachable* for time $t > 0$ if there exists $x \in \mathbb{R}^n$ with

$$t \leq |x_0 - x| = d(x). \tag{16}$$

The set of all backwards reachable points for time t is denoted by R_t.

See Figure 4a for an illustration of the set R_t: The point $x_0 \in R_t$ is shown together with a possible $x \in \Gamma_t$ that fulfils (16). Note that only the red part of Γ_0 is backwards reachable. Thus, we see that R_t is actually more regular than Γ_0 itself. In particular, Ω_0 has always uniform upper density on R_t. To understand why this must be the case, take a look at Figure 4b: Whenever x_0 and x are as indicated, the ball $B_t\left(x\right)$ must be disjoint to $\overline{\Omega}_0$ since otherwise $d(x) < t$ would be the case. Thus, the volume of $B_t\left(x\right) \cap B_t\left(x_0\right)$ can never be part of Ω_0, which implies an upper bound for the density of Ω_0 at x_0. (For the shown situation, the density is actually $1/2$. The maximal possible density would be achieved if also the light grey area were part of Ω_0.) Let us formalise this argument:



(a) The set R_t of Definition 5. (b) The main argument in the proof of Lemma 6.

Figure 4: The backwards reachable set and its regularity with respect to uniform upper density. The dark grey region is Ω_0. The point x_0 is on $R_t \subset \Gamma_0$, with $x \in \Gamma_t$ such that (16) holds.

Lemma 6. *Let $t > 0$ and R_t be the backwards reachable set for time t. Then Ω_0 has (t, c)-uniform upper density on R_t, where c is a dimensional constant.*

Proof. Let $e \in \mathbb{R}^n$ be arbitrary with $|e| = 1$. We define

$$0 < c = \frac{\operatorname{vol}(B_1(0) \cap B_1(e))}{\operatorname{vol}(B_1(0))} < 1.$$

Now choose $x_0 \in R_t$ and $\tau \leq t$. We have to show that (14) holds for $B_\tau(x_0)$ with the defined c. By Definition 5, there exists $x \in \mathbb{R}^n$ such that $\tau \leq t \leq |x_0 - x| = d(x)$. We can assume, without loss of generality, that $|x_0 - x| = \tau$. Considering Figure 4b, this implies $B_\tau(x) \cap \Omega_0 = \emptyset$. Hence:

$$\frac{\operatorname{vol}(B_\tau(x_0) \cap \Omega_0)}{\operatorname{vol}(B_\tau(x_0))} = 1 - \frac{\operatorname{vol}(B_\tau(x_0) \setminus \Omega_0)}{\operatorname{vol}(B_\tau(x_0))} \leq 1 - \frac{\operatorname{vol}(B_\tau(x_0) \cap B_\tau(x))}{\operatorname{vol}(B_\tau(x_0))} = 1 - c$$

□

Another important observation is that the backwards reachable set is already sufficient for the construction of the newly created volume U_t. This allows us to restrict our considerations to the more regular R_t instead of Γ_0 itself later on.

Lemma 7. *For $0 < s < t$, $R_t \subset R_s$. Furthermore,*

$$U_t \setminus U_s \subset \bigcup_{x_0 \in R_s} B_t(x_0).$$

Proof. The inclusion $R_t \subset R_s$ is immediately clear from Definition 5. Pick $x \in U_t \setminus U_s$ arbitrarily. By Lemma 4 we can find $x_0 \in \Gamma_0$ with $d(x) = |x_0 - x|$. Moreover, $x \notin U_s$ implies that $d(x) \geq s$, so that $x_0 \in R_s$. Similarly, $x \in U_t$ yields $d(x) < t$ and thus $x \in B_t(x_0)$. □

With this result, all preparations are in place and we can proceed to the actual estimate of $\operatorname{vol}(U_t)$. This is done in two steps: First, we estimate $\operatorname{vol}(U_{2t} \setminus U_t)$. The regularity of the backwards reachable set with respect to uniform upper density of Ω_0 can be used for this situation. Afterwards, we build the union of a sequence of such strips in order to get $\operatorname{vol}(U_t)$ itself.

Lemma 8. *Assume that Ω_0 has (t_0, c)-uniform lower density on Γ_0. Then there exists a dimensional constant C such that*

$$\operatorname{vol}(U_{2t} \setminus U_t) \leq C \left(1 + \frac{1}{c}\right)^{\frac{n-1}{n}} t \cdot P(\Omega_0)$$

holds for all $t \in (0, t_0)$.

Proof. According to Lemma 6, we know that Ω_0 has (t, c')-uniform upper density on R_t with some dimensional c'. Since it has uniform lower density per assumption, it has (t, c'')-uniform density (both upper and lower) with $c'' = \min(c, c')$. Furthermore, note that

$$\frac{1}{c''} = \frac{1}{\min(c, c')} \leq \frac{1}{c} + \frac{1}{c'}.$$

Thus Lemma 5 implies that

$$t^n \leq C' \left(\frac{1}{c} + \frac{1}{c'}\right)^{\frac{n-1}{n}} t \cdot \mathcal{H}^{n-1}(B_t(x_0) \cap \mathcal{F}\Omega_0)$$

for all $x_0 \in R_t$ with some dimensional C'. Taking it even further, this yields also

$$\operatorname{vol}\left(\overline{B_{10t}(x_0)}\right) = 10^n \cdot \operatorname{vol}\left(\overline{B_t(x_0)}\right) \leq C \left(1 + \frac{1}{c}\right)^{\frac{n-1}{n}} t \cdot \mathcal{H}^{n-1}(B_t(x_0) \cap \mathcal{F}\Omega_0) \tag{17}$$

for yet another dimensional constant C.

Making use of Lemma 7, we know that

$$U_{2t} \setminus U_t \subset \bigcup_{x_0 \in R_t} B_{2t}(x_0).$$

With Vitali's covering theorem (see, again, Theorem 1 on page 27 of [11]), we can construct $X \subset R_t$ at most countable such that the sets $B_{2t}(x_0)$ are disjoint for $x_0 \in X$, but still

$$U_{2t} \setminus U_t \subset \bigcup_{x_0 \in X} \overline{B_{10t}(x_0)}.$$

Taking the measure on both sides of this inclusion and using (17), we finally find

$$\begin{aligned}\operatorname{vol}(U_{2t} \setminus U_t) &\leq \sum_{x_0 \in X} \operatorname{vol}\left(\overline{B_{10t}(x_0)}\right) \leq C \left(1 + \frac{1}{c}\right)^{\frac{n-1}{n}} t \cdot \sum_{x_0 \in X} \mathcal{H}^{n-1}(B_t(x_0) \cap \mathcal{F}\Omega_0) \\ &\leq C \left(1 + \frac{1}{c}\right)^{\frac{n-1}{n}} t \cdot \mathcal{H}^{n-1}(\mathcal{F}\Omega_0) = C \left(1 + \frac{1}{c}\right)^{\frac{n-1}{n}} t \cdot P(\Omega_0).\end{aligned}$$

The simplification of the sum is justified because all sets $B_t(x_0)$ are disjoint. □

Theorem 4. *Let Ω_0 have (t_0, c)-uniform lower density on Γ_0. Then*

$$\frac{\operatorname{vol}(U_t)}{t} \leq C \left(1 + \frac{1}{c}\right)^{\frac{n-1}{n}} P(\Omega_0)$$

for all $t \in (0, t_0)$ and a dimensional constant C.

Proof. Let $t \in (0, t_0)$ be given. Then the disjoint telescopic decomposition

$$U_t = \left(U_t \setminus U_{t/2}\right) \cup \left(U_{t/2} \setminus U_{t/4}\right) \cup \dots = \bigcup_{k=1}^{\infty} \left(U_{2t/2^k} \setminus U_{t/2^k}\right)$$

holds. Together with Lemma 8 this yields

$$\operatorname{vol}(U_t) = \sum_{k=1}^{\infty} \operatorname{vol}\left(U_{2t/2^k} \setminus U_{t/2^k}\right) \leq C \left(1 + \frac{1}{c}\right)^{\frac{n-1}{n}} P(\Omega_0) \cdot \sum_{k=1}^{\infty} \frac{t}{2^k} = C \left(1 + \frac{1}{c}\right)^{\frac{n-1}{n}} t \cdot P(\Omega_0).$$

This finishes the proof. □

When we combine Theorem 4 with Theorem 3, we finally get a uniform bound for $\mathcal{H}^{n-1}(\Gamma_t)$. This result is very similar to Theorem 6 in [7], but note that it holds *for all* $t \geq 0$ and not just for almost all:

Corollary 2. *Assume that Ω_0 has (t_0, c)-uniform lower density on Γ_0 and that Ω_0 is bounded. In particular, let $\Omega_0 \subset B_R(0)$ for some $R > 0$. Then*

$$\mathcal{H}^{n-1}(\Gamma_t) \leq C \cdot \left(1 + P(\Omega_0) + t^{n-1}\right) \tag{18}$$

for all $t \geq 0$. The constant C depends only on n, t_0, c and R but no other properties of Ω_0.

Proof. Note that the situation is clear for $t = 0$ as long as we choose $C \geq 1$. From Theorem 4, we know that $\operatorname{vol}(U_t)/t \leq C' P(\Omega_0)$ for all $t \in (0, t_0)$. Furthermore, since $\Omega_0 \subset B_R(0)$, note that $\Omega_t \subset B_{R+t}(0)$. Thus, for $t \geq t_0$,

$$\operatorname{vol}(U_t) \leq \omega_n (R+t)^n \leq C''(1+t^n) \quad \Rightarrow \quad \frac{\operatorname{vol}(U_t)}{t} \leq C'' \left(\frac{1}{t} + t^{n-1} \right) \leq C''' \left(1 + t^{n-1}\right).$$

The claim now follows from Theorem 3, if we combine both estimates for $\operatorname{vol}(U_t)/t$. □

4.3 Geometric Regularity Properties in the Literature

The main ingredient for the results in the previous Subsection 4.2 is a particular geometric property of the initial set Ω_0, namely uniform density from Definition 4. As pointed out above, this notion has been used already by others to achieve similar results (see [2] and [7]). We are not aware of any applications in a broader context, though. Thus, it makes sense to put it into perspective with similar geometric properties that are more established in the literature and more widely used. In particular, a variety of so-called *(uniform) segment* and *cone properties* is often used to characterise geometric regularity of sets. For a thorough introduction, see Section 2.6 of [9].

Since uniformity plays an important role for the results of Subsection 4.2, it makes only sense to consider the *uniform* variants of those segment properties. (All non-uniform properties are fulfilled by the example developed in Section 2, since it is constructed only from circles.) Furthermore, the uniform (fat) segment property alone also provides very little regularity. For instance, a cusp satisfies it while it clearly does not have uniform lower density. Thus, let us focus on the *uniform cone property.* For convenience, we recall Definition 6.3 on page 115 of [9]:

Definition 6. For $x_0, x \in \mathbb{R}^n$ and $\phi \in [0, \pi/2]$, define the open cone

$$C_\phi(x_0, x) = \left\{ y \in \mathbb{R}^n \;\middle|\; |x_0 - y|\,|x_0 - x| \cdot \cos\phi < (y - x_0) \cdot (x - x_0) < |x_0 - x|^2 \right\}.$$

This is similar to the sector $S_\phi(x_0, x)$ of Definition 1 studied above, but it describes a cone with flat base, i. e., without a spherical cap.

Now, let $\Omega \subset \mathbb{R}^n$ be open. We say that Ω satisfies the *uniform cone property* if there exist $t > 0$, $\phi \in (0, \pi/2)$ and $\rho > 0$ such that for all $x_0 \in \partial\Omega$ there is $x \in \mathbb{R}^n$ with $|x_0 - x| = t$ and

$$x + d \in \overline{\Omega} \quad \Rightarrow \quad C_\phi(x_0 + d, x + d) \subset \Omega$$

for all $d \in B_\rho(0)$.

Since the uniform cone property ensures for each boundary point the existence of a cone that is entirely contained in Ω, we can use this cone's volume as a lower bound on the density of Ω. Thus, the uniform cone property is a stronger condition than uniform lower density:

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ satisfy the uniform cone property with t and ϕ as in Definition 6. Then Ω has $(t, r(\phi))$-uniform lower density on $\partial\Omega$. Similarly, if $\mathbb{R}^n \setminus \overline{\Omega}$ has the uniform cone property with these constants, then Ω has $(t, r(\phi))$-uniform upper density.*

Proof. Let $x_0 \in \partial\Omega$ be given. According to Definition 6, there exists $x \in \mathbb{R}^n$ with $|x_0 - x| = t$ such that $C_\phi(x_0, x) \subset \Omega$. Note that $S_\phi(x_0, x) \subset C_\phi(x_0, x)$ since

$$|x_0 - y| < |x_0 - x| \quad \Rightarrow \quad (y - x_0) \cdot (x - x_0) \leq |y - x_0| \cdot |x - x_0| < |x_0 - x|^2.$$

Thus, for each $\tau \in (0, t)$, clearly

$$B_\tau(x_0) \cap S_\phi(x_0, x) \subset B_\tau(x_0) \cap \Omega.$$

Hence, we can estimate

$$\operatorname{vol}(B_\tau(x_0) \cap \Omega) \geq \operatorname{vol}(B_\tau(x_0) \cap S_\phi(x_0, x)) = r(\phi) \cdot \operatorname{vol}(B_\tau(x_0))$$

based on (6). This shows the claim. The proof for uniform upper density works analogously. □

Another concept related to our definition of uniform lower density are sets with *finite density perimeter* as defined in [4] and Subsection 3.1 of [8]:

Definition 7. Let $\Omega \subset \mathbb{R}^n$ be open and $h > 0$. The *h-density perimeter* of Ω is then defined as

$$P_h(\Omega) = \sup_{0 < \epsilon < h} \frac{\operatorname{vol}(V_\epsilon(\partial\Omega))}{2\epsilon}, \tag{19}$$

where $V_\epsilon(\partial\Omega)$ is the *ϵ-envelope* of $\partial\Omega$:

$$V_\epsilon(\partial\Omega) = \bigcup_{x \in \partial\Omega} B_\epsilon(x) = \{x \in \mathbb{R}^n \mid d_{\partial\Omega}(x) < \epsilon\}$$

If $P_h(\Omega)$ is finite, we call Ω a set of *finite h-density perimeter.*

This can be interpreted as a relaxation of the $(n-1)$-dimensional Minkowski content (see, for instance, 3.2.37 in [12]). To be precise, the Minkowski content results if the supremum in (19) is replaced by the limit $\epsilon \to 0^+$. It is easy to see that $V_\epsilon(\Omega_0)$ is related to the newly created volume U_ϵ of Definition 3: The set U_ϵ is the part of $V_\epsilon(\Omega_0)$ which is outside of $\overline{\Omega}_0$. Hence, an argument similar to the proof of Theorem 4 can be applied to show that *uniform density implies finite density perimeter*.

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